Writing to Learn in Mathematics

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The majority of people, mathematicians included, think that writing out formulas is exactly what we call writing in mathematics. I was guilty of the same preconceptions before I started to work with the Writing Across the Curriculum Project at Medgar Evers College (WAC @ MEC). The definition of writing to learn that we use at MEC helped me come up with the idea that served as the basic principle for my further experiments and conclusions as I implemented writing to learn in mathematics. Our definition of WAC @ MEC is this:

> We define writing across the curriculum as a means to connect writing to learning in all content areas. We define writing as the process through which students think on paper, explore ideas, raise questions, attempt solutions, uncover processes, build and defend arguments, brainstorm, introspect, and figure out what's going on. We define all of these as thinking. Writing to learn across the curriculum provides a potent way for students to exercise their own voices as well as to take on new voices which represent their knowledge of the content and <u>the language of the discipline they are learning</u> (Lester, et. al, 2000, p.4).

The words that I have underlined gave me the idea of what the concept of writing to learn in mathematics should be–learning the new language of the new discipline.

Why Do We Have to Write in Mathematics?

My long experience as a college professor, as well as my being a foreign professor teaching in the United States and having English as a second language, has allowed me to notice a similarity between learning mathematics and learning a foreign language. Mathematics, just as any other subject, has its own very specific language in which every word is rigorously defined. For example, a common word, like "between," when used in geometry obtains a very precise meaning: We say that point C is **between** points A and B if and only if all three points are on the same line and AC + CB = AB. Often the words are defined in terms of formulas; this is the nature of mathematics. But at the same time, all formulas have verbal meanings that are analogous to the translation from one language to another and work as a glossary. For example, the well-known formula $a^2 + b^2 = c^2$ has an equivalent translation that can be read as "the sum of the squares of the legs of the right triangle is equal to the square of the hypotenuse."

In narrative language, we cannot use a word correctly in context if we do not know what it means, even though we might know the word and can spell it. The same is true for mathematics: Even knowing a phrase, for example, "an increasing function," but being unable to explain the concept means we cannot solve a simple problem of finding the intervals of increase and decrease of a given function.

It is impossible to conceive of learning a second language without writing. ESL professors say: "What ESL students need [are] multiple opportunities to use language and write-to-learn" (Zamel,V., 1995, p.261). One type of using language to learn occurs in exercises in translation from one language to another in writing. The same should be done in mathematics: translation of the formulas (explanation of the formulas) in narrative English (or any other language) and vice versa needs to be made in order to understand the mathematics involved. This translation should be done in writing for these reasons:

1) Writing allows students to organize their thoughts. James Britton, in his 1970 book *Language and Learning*, argues that language is central to learning because through language we "organize our representation of the world" (Russell, D.R., 1990, p. 277). A written statement can be revised and corrected, and "revising or *re*-vision means taking another look, to see again what has already been seen, but this time from a different perspective" (Mayher, J.S., Lester, N., Pradl, G.M., 1983, p.43).
2) Written means visualized, and it is easier to see a mistake rather than to hear it. When we speak we compose. When we write we compose even better because we can manipulate our compositions on paper, in addition to holding them in our heads. We can re-view them, revise them and re-write them because they are now visible and concrete (Fulwiler, T.,1983, p. 279).
3) Many people have dominant visual and motor memory, which means they learn written words more easily than words they just hear or read.
4) No other class assignment gives such complete feedback as a written assignment, because usually in class not everybody speaks up and asks questions and often the level of students' misunderstanding is shown only on the test when students are already beyond help.

Experimental Writing to Learn in Precalculus

For the implementation of all of these principles, I chose my Precalculus class. It was chosen for the experiment because in our college, Precalculus is the first serious course in mathematics that students majoring in the sciences must take. And science students were chosen because they understand the importance of mathematics in their education and more comfortably accept innovations in teaching than students for whom mathematics is a necessary burden they have to pass and forget. As a rule, students who take Precalculus have algebraic skills and know how to manipulate formulas, but have a very limited conceptual understanding of the subject, which is absolutely necessary for all upper level courses in mathematics.

While working on theoretical aspects of writing to learn, I used traditional methods of teaching and gave a traditional test. One of the test's problems was *Determine the intervals over which the function is increasing, decreasing, or constant*. The function was given in symbolic form. I did not ask for verbal explanations. Students were supposed to draw the graph of the function using a graphing utility, estimate x-coordinates of maximum and minimum points, and write all intervals on which the function increases in the form (x_{min},x_{max}) and all intervals on which the function decreases in the form (x_{max}, x_{min}). There were no intervals on which the function is constant.

Only half the students solved the problem correctly. Some of them did not know where to start; some of them found several x-coordinates of max and min points but did not know what to do with them. After the exam, I asked the students to describe the problem in their own words by prompting them with the following question: *What does it mean graphically that function increases or decreases on certain intervals?* I expected very simple and general explanations, like the following:

> *On the intervals where the function increases, the graph rises; on the intervals where the function decreases, the graph falls; in order to define the end points of the intervals where the function increases or decreases, find the x-coordinates of maximum and minimum points.*

The students' answers confirmed the grades they received: The ones who solved the problem correctly were able to explain the concept of increasing and decreasing function in their own words; the ones who could not solve the problem gave a variety of unacceptable written responses. Below, I introduce the whole spectrum of students' responses, from unacceptable to effective, in the language they used to answer the prompt. I want to point out that I did not pay attention to grammar, punctuation, or completeness of sentences, but focused only on the mathematical content, even though I was shocked by the lack of ability that English-speaking people had to express their thoughts in writing.

Students' Writing to Learn

As I pointed out above, students were to describe in their own words the graphical meaning of increasing or decreasing function. Student A's answer was:

> *The intervals increases and decreases is where the functions connect with x and the y is x or -x it is decreasing and where the y both increases in value after decreasing.*

This sentence does not make any sense, and it reveals a lack of understanding of the concept.

Student B remembered my explanation of using a computer for finding the intervals over which function increases or decreases, but she did not understand the definition and, as a result, she could not solve the problem. Her answer was:

> *We put the cross in the two points where the cross is up and down to get the x-intervals where it is increase and decrease.*

She understands what points you have to pay attention to, but does not know what to do with them.

At the same time, the students who solved the problem were able to explain the concept. Sometimes they did it in simple words, such as Student C:

> *Where function increases - as x-values increase the curve goes upwards. Where function decreases - as x-values increase the curve goes downward.*

This explanation could be written in more standard English, but it is clear that the student understands the idea of increasing and decreasing functions and, as the test showed, knows how to use it.

Student D gave a more detailed explanation:

> *To determine where the function decrease, I will go to the left side of the graph and I will take all the x-values where my function is decreasing, which is (-x,-2) . I stop by -2 because it is the last point where the function is decreasing in that part. After that function goes up which is mean the function is increasing.*

Again, the sentence is a little bit awkward in terms of standard English, but very precise in the understanding of the concept: the student understands that he has to move along the graph from left to right identifying x-coordinates of the maximum and minimum points. He uses the words "goes up" meaning "increases."

Other students' answers confirm the direct correspondence between perception of the definition, the ability to explain it, and the capability to solve problems.

I cannot help but mention one explanation, which is not only absolutely correct but also very poetic. Answering the question: *What is the maximum of a function?*; Student E. wrote:

> *[The] max of the function is where increase meets decrease.*

Rigorous mathematicians might argue the mathematical preciseness of that explanation, but its visual and imaginative accuracy is undoubted. One can see how the graph of the function rises to the maximum point and then falls down.

While covering the next chapter in the textbook and preparing students for the next test, I specified the basic definitions and procedures students had to know to be able to solve the problems. In the beginning of the classes, students were given the prompts to write definitions, explanations, and algorithms of problem-solving. One of the most important concepts in studying rational functions is the notion of vertical and horizontal asymptote. Students had to write the definitions of the vertical and horizontal asymptotes, like the one that follows:

> The line $x = a$ is a vertical asymptote for the graph of $y = f(x)$ if $f(x)$ either increases of decreases without

> bound as x approaches a from the right or from the left. The line $y = b$ is a horizontal asymptote for the graph of $y = f(x)$ if $f(x)$ approaches b as x increases without bound or as x decreases without bound.

And they had to describe the procedure for finding them:

> To find the vertical asymptote set the denominator of the function equal to zero, solve for x, then the line $x = a$, where a is a solution of the above mentioned equation, is a vertical asymptote. To find the horizontal asymptote, consider degrees of the numerator, m, and the denominator, n, and leading coefficient of the numerator, a_m, and the leading coefficient of the denominator, b_n. If $m < n$, the line $y = 0$ is a horizontal asymptote. If $m = n$, the line $y = a_m/b_n$ is a horizontal asymptote. If $m > n$, there are no horizontal asymptotes.

From reading students' papers, I found conceptual misunderstandings that I would never have thought of. Some students did not understand that an asymptote is a line; they thought that it was a number. Others thought that only coordinate axes could be the asymptotes. Some of them described the procedure for finding the asymptote correctly but did not know how to apply it. For example, student A's answer was:

> *Set the denominator equal to zero.*

That is a correct answer, but he wrote the wrong definition of the vertical asymptote itself as an answer to the prompt, *Write down the definition of the horizontal asymptote*, and so he does not know what to do with his answer, *denominator equal to zero*, and cannot draw a correct line.

Often students do not understand the difference between the definition and the procedure for finding the object. For example, answering the question about the definition of the horizontal asymptote, student E wrote:

> *Horizontal asymptote - if the degree of the equation*

> *are the same then the horizontal asymptote is the ratio of the coefficients of the leading terms. If the degree of the numerator is less then the denominator then the x-axis is the asymptote. If the degree of the numerator is greater then the denominator then there is no horizontal asymptote.*

This is a typical example of the confusion between a definition and the procedure: a horizontal asymptote is a horizontal line that satisfies to certain conditions, and in order to find an equation of that line we have to consider the relationship between degrees of the polynomials in the numerator and denominator of the function.

I had never had such complete information about each individual student's knowledge of the topic before a test until I began using writing to learn. I organized, as a discussion session, the review session on this topic. Students were very enthusiastic about discussing the subject and trying to find and correct mistakes their classmates made. Together, we made up correct definitions and procedures.

The test consisted of several problems where the students had to analyze given rational functions and sketch their graphs using information obtained in the analyses part. My disappointment was tremendous when I found out that many students described completely and correctly all the necessary steps in the analyses part, but eventually could not sketch the graph of the rational function, which was the final goal of the problem. That means that students cannot see the whole beneath the parts. They are able to solve simple problems such as finding the equations of the asymptotes or x- and y-intercepts of the function, but they are unable to sketch the graph of the function even though they have all the necessary information.

For the rest of the semester I continued to work in this class using the same scheme: (1) identifying the basic concepts of the topic studied; (2) making students respond in writing to short prompts, and (3) discussing their answers during the next class.

The final results were very encouraging: Almost half of the students received high grades; only two students failed, but I could, based on their writing to learn, predict this after the first few weeks of the semester. In comparison with my usual results in Precalculus classes—no more than 20 percent of high grades and no less than 20 percent of failing grades—this outcome looks wonderful. Moreover, at least four students wanted to start research in mathematics even though they did not complete the calculus sequence.

What I Learned: Possibilities and Challenges

No other form of class work or homework gives such powerful feedback as short written "low stakes" assignments (Elbow, P., 1997, p.5) in class, i.e. the prompts that are not graded. The professor can see very clearly what every individual student does not understand. Sometimes students make a mistake the professor would never dream of; for example, the student's misconception that asymptote is a number. It is always useful to talk in class about these kinds of rare mistakes; one can look at the problem from different points of view. "Low stakes" writing assignments help to organize individual work with students. The professor knows the strengths and weaknesses of each student and the ways each individual learns.

Students know that they will have writing assignments in class and take more effort to prepare for the lesson, since, psychologically, they take writing to learn more seriously than ordinary homework. Students usually are not ashamed if they do not submit homework, but they feel uncomfortable if they should submit a blank paper when they are supposed to respond in writing to a prompt. Discussion of papers in class (anonymous) and attempts to find and correct mistakes help students understand and learn the covered material. This kind of work enlivens the teaching and learning process and reduces the routine and monotony of lecturing.

At the same time, there are many professional problems in any serious implementation of "writing to learn." The prompts for low stakes writing should be very clearly formulated and not leave any ambiguities. Writing in mathematics is very unusual for the students and we do not want them to be more confused than they already may be. An unclearly stated prompt, like *Describe the step-by-step procedure of how to graph any polynomial function* in Precalculus led to incomplete answers. Student R answered this way:

> *1) find the zeros by factoring, 2) plot the points of the x-axis, 3) then find the y-intercept, 4) join the points with a smooth curve, 5) find the leading coefficient and determine the left hand and right hand behavior of the polynomial.*

Even when taking into consideration some minor mistakes (e.g. one can not always find zeros by factoring, no indication of how to join the points with a smooth curve, or right hand and left hand behavior depends not only on the leading coefficient but on the degree of the polynomial as well), it is obvious that Student R understands what he is talking about.

But my goal was obviously to get a more extensive explanation. Precisely, I wanted to know how the leading coefficient and degree of the polynomial influences the right hand and left hand behavior, how to find the x- and y-intercept, and how to define the behavior of the function between the consecutive zeros. It is apparent that I should have formulated my question more comprehensively: *Describe the step-by-step procedure of how to graph any polynomial function. Include a complete explanation of the Leading Coefficient Test, describe all specific points you need for the graphing, and explain how to find the behavior of the function between consecutive zeroes.* Learning from my own mistakes, I plan to make up focused assignment descriptions for the whole semester and discuss them with colleagues.

The necessity of reading many papers every day is time consuming. Writing appropriate comments takes even more time.

The comments should not include just the correction of mathematical mistakes, but rather responses aimed at stimulating students' thinking about resolving the problem in a different way. Let me emphasize here that I am not talking about grammar or spelling, only about factual mathematical mistakes. It is very hard to come up with a few general comments that one can use in different cases. Mistakes students make are unpredictable, and sometimes the only comment I really want to make is *Where have you been while we studied this topic?*

Here is an example of my comments on the student's answer to the question on the step-by-step procedure of how to graph any polynomial function. Student J wrote:

> *First identify the leading coefficient to determine whether it is odd or even. If the coefficient is odd and positive, the graph falls at negative infinity.*

My comment was *Oddness or evenness of the leading coefficient has nothing to do with the graph's behavior.* My intention was to identify what was wrong with the answer. But rather than giving the student the correct answer directly, I provided her with hints that would steer her in the right direction. I wanted her to recall that oddness or evenness of the degree of the polynomial rather than oddness or evenness of the leading coefficient plays an important role in the right and left hand behavior of the function.

It is very hard to find time for individual work with students. Very often students cannot come during office hours. To talk in class if there are more than fifteen students enrolled is simply impossible. The only remaining thing to do is to compile a list of the most commonly made mistakes and to discuss them in class. This reduces the number of students who require individual attention.

Beyond the usefulness that WAC has played in my classes, I cannot help but mention the impact it has had on me as a writer. Being a pure mathematician, I was not exposed to extensive writing even though I have written innumerable papers and a

Ph.D. thesis. In mathematics it is simply definitions and proofs with a few sentences in between. Add to this the lack of English as my native language, which is Russian, and you have a typical basic writer. I feel much more comfortable and confident working on my own papers after a year of WAC.

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